

Nonsmooth backgrounds in quantum field theory

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The one-loop renormalization in field theories can be formulated in terms of the heat kernel expansion. In this paper we calculate leading contributions of discontinuities of background fields and their derivatives to the heat kernel coefficients. These results are then used to estimate contributions of the discontinuities to the Casimir energy. Sign of such contribution is defined solely by the order of discontinuous derivative. We also discuss renormalization in the presence of singular (delta-function) potentials. We show that an independent surface tension counterterm is necessary. This observation seems to resolve some contradictions in previous calculations.

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I. INTRODUCTION

Recent years have seen much progress in understanding the Casimir effect [1, 2]. Although this effect is oftenly considered as a macroscopic manifestation of quantum fields, many ingredients of quantum field theory have not been yet reformulated to the Casimir framework. Renormalization is probably the most important example of such an ingredient. Early calculations of the Casimir force between rigid bodies were not affected by the renormalization problems since the divergent part of the Casimir energy is local (does not depend on the distance) and, therefore, does not contribute to the force. There are, however, many quantities of interest (like the Casimir stress) which do contain a divergent part and require some renormalization.

There exists a subtraction prescription [2, 3] which allows to obtain well-defined results for the Casimir energy even if it is divergent. This prescription consists in expanding the regularised Casimir energy in log-power series for large mass m of the fluctuating field and subtracting all non-negative powers of m and $\log m$ terms. A similar prescription is being used in quantum field theory in curved space [4]. In two dimensional scalar theories it is equivalent to the “no tadpole” condition [5]. Unfortunately, no such statement is known for other models or other dimensions. Besides, this large mass prescription is not applicable for massless fields.

In order to remove ambiguities in the Casimir energy, it was suggested just a couple of years ago [6, 7] to perform the calculations in renormalizable theories. This direction attracted a lot of interest, but the results of different groups are still contradicting (cf. [8, 9] and references therein).

The main difficulty in applying the renormalization procedure of quantum field theory to the Casimir-type problems is that the latter one usually assume singular backgrounds (which may be penetrable walls, or boundary conditions, or just non-smooth potentials). This motivated us to study quantum field theory (in the one-loop approximation) in the presence of non-smooth background fields. Namely, we consider the case when p th derivative of the background field jumps on a surface Σ of co-dimension one.

In this paper we use the zeta function regularization and the heat kernel methods (both are sketched in sec. II). In this regularization, the one-loop divergences are expressed in terms of the heat kernel coefficients a_k . These coefficients are analysed in sec. III for non-smooth potentials. In sec. III A we calculate the linear order in the potential and find how the localised heat kernel is modified by the presence of singularities. Somewhat surprisingly, the global heat kernel (which defines the counterterms) is not sensitive to the singularities in this order. The quadratic order in the potential is studied in sec. III B, where we calculate leading contributions to global heat kernel. There we prove a “folk theorem” that if standard “smooth” expression for a_k is divergent due to the singularity, a non-zero surface contribution to a_{k-1} should appear. These results allow us to calculate the leading contribution of the singularity to the Casimir energy (sec. III C). Then, in sec. IV we discuss renormalization of theories with delta-function singularities.

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II. THE REGULARIZATION

In this section we sketch basic technical results on the zeta-function regularization and the heat kernel. For a more detailed introduction one may consult the monographs [11] or the review [12]. The reader may also consult a recent paper by Fulling [13] which deals with applications of the heat kernel technique to the Casimir energy calculations.

To simplify our discussion in this paper we consider scalar field theories only. The classical action reads

$$\mathcal{L} = \int_M d^n x \left(\frac{1}{2} (\nabla \varphi)^2 + U(\varphi) \right). \quad (1)$$

We suppose that the manifold M is flat, so that it is either torus or \mathbb{R}^n . For $M = \mathbb{R}^n$ one has to assume some fall-off conditions on the fields to obtain convergent integrals. We assume also that the metric on M has Euclidean signature. We use the background field formalism, so that we split $\varphi = \Phi + \phi$ where Φ is a background field, and ϕ is a quantum fluctuation. Note, that the background field Φ may also include a part which describes “external conditions” (like, e.g., boundary conditions or domain walls or membranes). To calculate the one-loop effective action one should expand the action (1) about the background field Φ keeping the quadratic order terms in fluctuations only.

$$\mathcal{L}_2 = \frac{1}{2} \int_M d^n x \phi D[\Phi] \phi, \quad (2)$$

where $D[\Phi]$ is an operator of Laplace type,

$$D = -(\nabla^2 + E), \quad E = -U''(\Phi). \quad (3)$$

For a multicomponent φ the potential E is matrix-valued.

Formal path integration over ϕ leads to the following result for the one-loop effective action

$$W = \frac{1}{2} \ln \det(D). \quad (4)$$

Right hand side of (4) is divergent and has to be regularised. To this end we use an integral representation for the determinant

$$W_s = -\frac{1}{2} \mu^{2s} \int_0^\infty \frac{dt}{t^{1-s}} K(t, D), \quad (5)$$

with a regularization parameter s (the regularization is removed in the limit $s \rightarrow 0$). The heat kernel $K(t, D)$ is defined as a functional trace,

$$K(t, D) = \text{Tr}_{L^2}(e^{-tD}). \quad (6)$$

Let us define the zeta function of D by the equation:

$$\zeta(s, D) = \text{Tr}_{L^2}(D^{-s}). \quad (7)$$

Now we can rewrite W_s in terms of the zeta function

$$W_s = -\frac{1}{2} \mu^{2s} \Gamma(s) \zeta(s, D). \quad (8)$$

This procedure is called the zeta-function regularization [10].

The regularised effective action (8) has a pole at $s = 0$:

$$W_s = -\frac{1}{2} \left(\frac{1}{s} - \gamma_E + \ln \mu^2 \right) \zeta(0, D) - \frac{1}{2} \zeta'(0, D), \quad (9)$$

where γ_E is the Euler constant.

There is an asymptotic series as $t \rightarrow +0$

$$K(f, t, D) = \text{Tr}_{L^2}(f \exp(-tD)) \cong \sum_{k \geq 0} t^{(k-n)/2} a_k(f, D). \quad (10)$$

(here we have introduced a heat kernel smeared with a smooth function f . It is related to the unsmeared kernel (6) by $K(t, D) = K(1, t, D)$.) If E is smooth and M has no boundary, all odd numbered coefficients in (10) vanish, $a_{2k+1} = 0$.

There is an important relation

$$\zeta(0, D) = a_n(1, D) \quad (11)$$

which tells that the one-loop divergences are defined by the heat kernel asymptotics. We remind that n is dimension of the underlying manifold M . Besides that, the heat kernel expansion defines short-distance behaviour of the propagator and the large mass expansion of the effective action.

We shall use also a bi-local kernel $K(x, y; t)$ of e^{-tD} . It is related to the smeared heat kernel by means of the equation

$$K(f, D, t) = \int_M dx \operatorname{tr} f(x) K(x, x; t). \quad (12)$$

Here tr denotes a matrix trace over all discrete indices if they are in the model. It should be distinguished from Tr_{L^2} which is a functional trace in the space of square integrable functions.

Let us consider a surface Σ of co-dimension one in M . We take a background field Φ which is smooth everywhere except for Σ where its p th normal derivative has a discontinuity. The potential E shall then possess the same property. At least locally, Σ divides M into two parts M^+ and M^- . Let ν^+ and ν^- be unit normal vectors to Σ pointing inside M^+ and M^- respectively. Since M is smooth, $\nu^- = -\nu^+$, but this may be not true for more general geometries. Let $E^{(k)+}$ (respectively $E^{(k)-}$) be a limit of k th derivative w.r.t. ν^+ (resp. ν^-) of $E(x)$ as $x \rightarrow \Sigma$ from the M^+ (resp. M^-) side. If the k th derivative is continuous, $E^{(k)+} = (-1)^k E^{(k)-}$. For example, if $M = \mathbb{R}^1$ and $\Sigma = \{x = 0\}$, then $M^\pm = \mathbb{R}_\pm$, $E^{(1)\pm} = \pm \partial_x E$.

III. HEAT KERNEL FOR NON-SMOOTH POTENTIALS

In this section we restrict ourselves to the case of flat Σ . For a non-flat Σ the heat kernel expansion should contain additional terms with extrinsic curvature of Σ . Such terms have larger canonical (mass) dimensions than the terms considered below and shall, therefore, contribute to higher heat kernel coefficients.

A. Local heat kernel in the linear order

To analyse the heat kernel asymptotics we use the perturbative expansion [16] (see also [12] for a short overview). The exponent $\exp(-tD) = \exp(t(\Delta + E))$ with $\Delta = \partial_\mu^2$ can be expanded in a power series in E :

$$e^{-tD} = e^{t\Delta} + \int_0^t ds e^{(t-s)\Delta} E e^{s\Delta} + \int_0^t ds_2 \int_0^{s_2} ds_1 e^{(t-s_2)\Delta} E e^{(s_2-s_1)\Delta} E e^{s_1\Delta} + \dots \quad (13)$$

This expansion is purely algebraic. Each order of E in (13) is given by a convergent integral if E is smooth or has a singularity located on a surface of co-dimension one (this can be a discontinuity of a derivative or even a delta-function singularity). In the case of δ -singularities on a submanifold of co-dimension two or higher, there might be problems with the convergence. More careful estimates can be found in Ref. [17].

Here we analyse the heat kernel expansion to the linear order in E . Let f be a smooth function on M . Let $K_0(x, y, t)$ be the heat kernel for $E = 0$. Then to this order

$$K(f, E, t) = K(f, 0, t) - \int_0^t d\tau \int_M dx \int_M dy f(x) K_0(x, y, \tau - t) E(y) K_0(y, x, \tau) + \dots \quad (14)$$

Obviously, the second term in (14) is symmetric w.r.t. exchanging the role of E and f . Therefore, it can be interpreted as a linear order term of the heat kernel with the potential f and the smearing function E . This term then reads:

$$\int_M dx \tilde{K}_1(x, x, t) E(x) \quad (15)$$

where \tilde{K}_1 is the heat kernel with the potential $f(x)$ at the linear order in f . The crucial point is that f is smooth. If we neglect all curvatures, then local heat kernel coefficients corresponding to \tilde{K} are:

$$\tilde{a}_{2k}(x) = (4\pi)^{-n/2} \alpha_2(k) \Delta^{k-1} f(x) \quad (16)$$

where [14]:

$$\alpha_2(k) = \frac{2k!}{(2k)!} \quad (17)$$

The heat kernel coefficients we are interested in read:

$$a_{2k} = (4\pi)^{-n/2} \alpha_2(k) \int_M dx E(x) \Delta^{k-1} f(x) \quad (18)$$

Odd-numbered coefficients are all zero. Formula (18) can be checked for $k = 1, 2$ [15].

An interesting observation regarding (18) is that all these coefficients vanish for $f = 1$, i.e. one loop divergences and the large mass expansion in the effective action are not affected by non-smoothness of the potential (to the linear order studied in this section).

B. Global heat kernel to quadratic order

In this section we study the heat trace asymptotics in the order E^2 and prove for this case a statement which existed in the folklore for many years: If due to a singularity of the background the volume contribution to a_{2k} diverges, the coefficient a_{2k-1} should have a non-zero surface contribution. This statement is based on particular case calculations for scalar backgrounds [3, 18] and for singular magnetic fields [19], and on analytic calculations of [15, 17]. A similar conclusion for dielectric problems follows from the analysis of [3]. In the present paper we give explicit expressions for the leading contribution to the heat kernel expansion from discontinuities of derivatives of the potential of arbitrary order.

We start with analysing relevant volume and surface invariants. For a smooth potential E there is an invariant $(\nabla^{p+1} E)^2$ (which means $(\nabla^\mu \Delta^{p/2} E)^2$ for p even and $(\Delta^{(p+1)/2} E)^2$ for p odd). It has dimension $(2p + 6)$ and can contribute to the coefficient a_{2p+6} :

$$a_{2p+6} \simeq (4\pi)^{-n/2} \beta(p) \text{tr} \int_M dx (\nabla^{p+1} E)^2. \quad (19)$$

If p th derivative of E is discontinuous the expression (19) contains a delta-function squared and is, therefore, meaningless. Then we expect that the following heat kernel coefficient appears

$$a_{2p+5} \simeq (4\pi)^{-(n-1)/2} \gamma(p) \text{tr} \int_\Sigma dx (E^{(p)+} - (-1)^p E^{(p)-})^2. \quad (20)$$

No other invariant of the same dimension can appear. For example, $(E^{(p)+} - (-1)^p E^{(p)-})(E^{(p)+} + (-1)^p E^{(p)-})$ changes sign if one exchanges the role M^+ and M^- , $(E^{(p)+} + (-1)^p E^{(p)-})^2$ would give a non-zero a_{2p+5} even for smooth potentials. According to the general theory both $\beta(p)$ and $\gamma(p)$ do not depend on n . We have to make sure that they are non-zero.

We use again the perturbative expansion (13). The heat trace can be also expanded,

$$K(t, D) = \text{Tr} (e^{-tD}) = \sum_{j=0}^{\infty} K_j(t), \quad (21)$$

where K_j contains the j th power of E . Localised version of K_1 has been studied above. In the next order we have:

$$\begin{aligned} K_2(t) &= \text{Tr} \left(\int_0^t ds_2 \int_0^{s_2} ds_1 e^{(t-s_2)\Delta} E e^{(s_2-s_1)\Delta} E e^{s_1\Delta} \right) \\ &= \text{tr} \int_M dy \int_M dz \int_0^t ds_2 \int_0^{s_2} ds_1 K_0(z, y; t - s_2 + s_1) E(y) \\ &\quad \times K_0(y, z; s_2 - s_1) E(z). \end{aligned} \quad (22)$$

To derive (22) we used cyclic property of the functional trace in order to combine $e^{(t-s_2)\Delta}$ with $e^{s_1\Delta}$.

Since the coefficients $\beta(p)$ and $\gamma(p)$ are universal constants we can use some particularly simple model to calculate their values. We take $M = \mathbb{R}$, $E(x < 0) = 0$, $E(x)$ smooth for $x > 0$ and decreases exponentially fast as $x \rightarrow +\infty$. We also suppose that first $p - 1$ derivatives are continuous at $x = 0$. Then

$$K_0(x, y; t) = (4\pi t)^{-1/2} \exp\left(-\frac{(x-y)^2}{4t}\right) \quad (23)$$

We adopt the strategy of [20]. After removing a redundant integration (22) takes the form:

$$K_2(t) = \frac{t}{8\pi} \int_0^\infty dx \int_0^\infty dy \int_0^t d\tau \frac{\exp\left[-(x-y)^2 \left(\frac{1}{4\tau} + \frac{1}{4(t-\tau)}\right)\right]}{\sqrt{\tau(t-\tau)}} E(x)E(y). \quad (24)$$

Next we integrate over τ to obtain

$$K_2(x, y; t) = \frac{t}{8} \int_0^\infty dx \int_0^\infty dy \operatorname{erfc}\left[\frac{|x-y|}{\sqrt{t}}\right] E(x)E(y). \quad (25)$$

Let us change the variables

$$\begin{aligned} x &= z + r\sqrt{t}, & y &= z & \text{for } x > y \\ y &= z + r\sqrt{t}, & x &= z & \text{for } x < y. \end{aligned} \quad (26)$$

In both cases $k, r \in [0, +\infty[$. Then

$$K_2(x, y; t) = \frac{t^{3/2}}{4} \int_0^\infty dz \int_0^\infty dr \operatorname{erfc}(r) E(z)E(z + r\sqrt{t}) \quad (27)$$

The $t \rightarrow 0$ asymptotic expansion is now performed by using the following formula:

$$\int_0^\infty f(r\sqrt{t}) \operatorname{erfc}(r) dr \simeq \sum_{n=0}^\infty t^{n/2} \frac{\Gamma(1 + \frac{n}{2})}{(n+1)! \sqrt{\pi}} f^{(n)}(0) \quad (28)$$

Now we only have to pick up relevant terms in the expansion. The term with $n = 2(p+1)$ reads:

$$\frac{t^{p+3}}{(4\pi t)^{1/2}} \frac{1}{2} \frac{2k!}{(2k)!} \int_0^\infty dz E(z) \partial_z^{2(p+1)} E(z), \quad (29)$$

which is consistent with (19) for smooth potentials if

$$\beta(p) = (-1)^{p+1} \frac{1}{2} \alpha_2(p+2). \quad (30)$$

α_2 is given by (17) above. Equation (30) follows easily from [14] and can be used as a consistency check.

The term with $n = 2p+1$ has the form:

$$t^{p+2} 4^{-(p+2)} \int_0^\infty dz E(z) \partial_z^{2p+1} E(z). \quad (31)$$

If first $p - 1$ derivatives of E vanish at $z = 0$, this result is consistent with (20) and gives

$$\gamma_p = (-1)^{p+1} 2^{-2p-5}. \quad (32)$$

We see, that both constants γ and β are non-zero.

Our calculation also confirms that as long as general “smooth” formulae for the heat kernel coefficients give convergent integrals no modifications appear due to the singularities. In the present context this means that old non-modified formulae are valid for a_k with $k < 2p+5$.

C. Application: Casimir energy density at the singularity

Let us calculate leading contribution of non-smoothness of the potential to the vacuum energy. In the zeta function regularization the ground state energy is defined as

$$\mathcal{E}(s) = \frac{1}{2} \mu^{2s} \sum_k \epsilon_k^{1-2s}, \quad (33)$$

where the regularization parameter s should be taken zero after the calculations, ϵ_k are eigenfrequencies of elementary excitations defined as square root of eigenvalues of the Hamiltonian:

$$H = -\partial_i^2 + V(x) + m^2 = D + m^2. \quad (34)$$

We shall work in a $3+1$ dimensional theory. Therefore, the operator D is three dimensional. V is a static potential which is supposed to be non-negative. We can rewrite (33) through the zeta function of H ,

$$\mathcal{E}(s) = \frac{1}{2} \mu^{2s} \text{Tr} \left(H^{\frac{1}{2}-s} \right) = \frac{1}{2} \mu^{2s} \zeta \left(s - \frac{1}{2}, H \right) \quad (35)$$

The zeta function, in turn, can be expressed through the heat kernel:

$$\zeta \left(s - \frac{1}{2}, H \right) = \frac{1}{\Gamma(s - \frac{1}{2})} \int_0^\infty dt t^{s-3/2} K(t, H) \quad (36)$$

and

$$K(t, H) = K(t, D) e^{-tm^2}. \quad (37)$$

It is easy to see that the large mass expansion of the vacuum energy is generated by the small t asymptotics of $K(t, D)$. Indeed, by substituting the heat kernel expansion in (35) with (36) and integrating over t we obtain:

$$\mathcal{E}(s) = \frac{1}{2} \mu^{2s} \frac{1}{\Gamma(s - \frac{1}{2})} \sum_k a_k(D) m^{-2s-k+4} \Gamma \left(s + \frac{k}{2} - 2 \right). \quad (38)$$

As expected, the terms with $k = 0, \dots, 4$ are divergent in the limit $s \rightarrow 0$.

We would like to separate contributions from non-smooth parts of the potential. If p th derivative jumps, the leading contribution comes from the coefficient a_{2p+5} in the heat kernel expansion. Note, that in four dimensions all contributions of these type are not divergent, so that we can put $s = 0$ already in (38). Then the term we are interested in reads

$$\mathcal{E}^{[p]} = -\frac{1}{4\sqrt{\pi}} a_{2p+5} m^{-2p-1} \Gamma \left(p + \frac{1}{2} \right). \quad (39)$$

Next we use (20) and (32) to obtain

$$\mathcal{E}^{[p]} = (-1)^p \frac{(2p-1)!!}{4\pi} 2^{-3p-7} m^{-2p-1} \int_\Sigma dx \left(\delta V^{(p)} \right)^2, \quad (40)$$

where $\delta V^{(p)}$ is discontinuity of p th normal derivative of V on Σ , according to our conventions $(-1)!! := 1$.

We see, that discontinuities in the potential itself and in its' even order derivatives tend to increase the vacuum energy, while discontinuities in odd order derivatives tend to decrease the vacuum energy. Of course, practically it may be not easy to separate contributions from continuous and discontinuous parts.

IV. RENORMALIZATION WITH SINGULARITIES

As we have already seen in the presence of singularities the heat kernel expansion is modified. This means that new counterterms may appear in quantum field theory at one loop. Non-smoothness of the potential modifies global heat kernel coefficients starting with a_5 (discontinuous potentials) or even higher (discontinuous derivatives). Therefore, they have no effect on counterterms in four dimensions. In order to be closer to physical applications we consider a

stronger singularity (delta-potentials) in four dimensions. This example is of particular interest because of extensive discussion in the literature (cf. recent works [8, 9] and references therein).

We start with some technical information. Consider an operator which has a singular part concentrated on the surface Σ :

$$D = -(\partial^2 + E(x) + v(x)\delta_\Sigma(x)). \quad (41)$$

For simplicity, we do not consider here any gauge fields and suppose that $E(x)$ is smooth. A mathematically correct formulation of spectral problem for operators with delta-like singularities yields a spectral problem for the regular part of D outside of Σ supplemented by matching conditions on Σ .

$$\begin{aligned} \phi^+|_\Sigma &= \phi^-|_\Sigma, \\ \left[\phi^{(1)+} + \phi^{(1)-} - v\phi \right]_\Sigma &= 0. \end{aligned} \quad (42)$$

We remind, that according to our notations $\phi^{(k)\pm}$ is a k th normal derivative of ϕ calculated on M^+ or M^- side of Σ . For a smooth ϕ : $\phi^{(1)+} = -\phi^{(1)-}$ (cf. sec. II).

The heat kernel coefficient a_4 which is responsible for one-loop divergences in four dimensions reads [17]:

$$a_4(D) = (4\pi)^{-n/2} \text{tr} \left[\int_M dx \frac{1}{2} E^2 + \int_\Sigma dx \left(\frac{1}{6} v^3 + Ev \right) \right] \quad (43)$$

Note, that no additional terms appear in (43) even if the surface Σ is curved. This implies that one needs the same counterterms on spherical surfaces as on flat ones.

Consider now an action containing a surface interaction term:

$$\mathcal{L} = \int_M d^4x \left(\frac{1}{2} (\nabla\varphi)^2 + \frac{m^2}{2} \varphi^2 + \frac{\lambda}{12} \varphi^4 + \Lambda \right) + \int_\Sigma d^3x \left(\sigma + \frac{\bar{\lambda}}{2} \varphi^2 \right). \quad (44)$$

Here m , λ , Λ , σ , and $\bar{\lambda}$ are some coupling constants. Next we introduce a continuous¹ background field Φ , $\varphi = \Phi + \phi$ and expand the action (44) up to quadratic order in quantum fluctuations ϕ . As a result, we obtain (2) where D is given by (41) and

$$E = -m^2 - \lambda\Phi^2, \quad v = -\lambda. \quad (45)$$

With the help of (9), (11), (43) and (45) we immediately obtain that the one loop divergences can be cancelled by the following counterterms:

$$\begin{aligned} \delta\Lambda &= \frac{1}{s} \frac{m^4}{32\pi^2} + \mathcal{O}(s^0), & \delta m^2 &= \frac{1}{s} \frac{\lambda m^2}{16\pi^2} + \mathcal{O}(s^0), & \delta\lambda &= \frac{1}{s} \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(s^0), \\ \delta\sigma &= \frac{1}{s} \frac{1}{32\pi^2} \left(-\frac{1}{6} \bar{\lambda}^3 + \bar{\lambda} m^2 \right) + \mathcal{O}(s^0), & \delta\bar{\lambda} &= \frac{1}{s} \frac{\bar{\lambda}\lambda}{32\pi^2} + \mathcal{O}(s^0), \end{aligned} \quad (46)$$

We see that the model (44) is renormalizable at least at one loop.

It is important that both constants Λ (the cosmological constant) and σ (the surface tension) must be present in the action to achieve renormalizability. One can consistently put $\Lambda = 0$ to all orders of perturbation theory since its *observed* value is negligible. The surface tension σ cannot be excluded on the same grounds. It has to be considered as an (experimental) input, as other coupling constants. To remove all ambiguities in the model one needs five independent normalisation conditions (including the “trivial” one $\Lambda = 0$) which should fix the $\mathcal{O}(s^0)$ terms in (46). To fix δm^2 and $\delta\lambda$ one can proceed as in non-singular theories. Namely, one can consider the effective action for a slowly varying background field Φ located far away from Σ . Then δm^2 and λ can be determined by prescribing certain values to the second and fourth derivatives of the effective action w.r.t. Φ at $\Phi = 0$. Physically this is equivalent to fixing position of the pole in the propagator of bosons and to fixing the value of the four-boson vertex for zero external momenta (both processes have to be considered very far from Σ). In principle, finite renormalization of $\bar{\lambda}$ can be fixed by relating the renormalized value of $\bar{\lambda}$ to some amplitudes of scattering of φ on Σ . It is not clear however which condition is the most convenient one. Unless all these conditions have been formulated one cannot give

¹ A stronger requirement that Φ is smooth would exclude classical solution of this model.

a scheme-independent meaning to the surface tension. This provides an alternative explanation to the contradictions reported in [9]. We should also note, that if one is interested only in the one-loop Casimir energy and stress on the background with $\Phi = 0$ (as in [9]), ambiguities in finite renormalization of m^2 , λ and $\bar{\lambda}$ are not important since they are of the order \hbar and do not enter the classical energy for $\Phi = 0$.

It is instructive to compare (44) to other models which appeared in the literature. The model considered by Milton [9] did not contain the self-interaction λ (which is not important) and the surface tension σ (which is very important for discussing the counterterms). The model of the MIT group [8] uses coupling to an external source. Although, the counterterm action suggested in [8] indeed allows to remove all divergences (at least as long as the source is smooth), the model shares a common difficulty of all models with non-dynamical external fields: the latter either have to be fixed artificially or require infinite number of “experiments” to be properly determined (if one “experiment” allows to determine one real number). Indeed, each coefficient in the Taylor expansion of the external source (or each form-factor) plays the role of an independent coupling.

There is a somewhat exotic example of a model with a surface interaction,

$$\tilde{\mathcal{L}} = \int_M d^4x \left(\frac{1}{2} (\nabla\varphi)^2 + \frac{\lambda}{12} \varphi^4 \right) + \int_\Sigma d^3x \frac{g}{2} \varphi^3, \quad (47)$$

which does not require any surface tension counterterms in the zeta-regularization at one loop. In other schemes (momentum cut-off, e.g.) such counterterms may appear also for (47).

V. CONCLUSIONS

In this paper we studied quantum field theory on non-smooth backgrounds. We have analysed leading contributions of the singularities to the heat kernel coefficients. We found that influence of the non-smoothness is rather mild: no new counterterms appear in dimension up to 4. We were also able to prove an important statement on the behaviour of the heat kernel expansion for singular backgrounds: “smooth” formulae for the heat kernel coefficients are valid as long as they are convergent. If the “smooth” expression for a_{2k} diverges for some k , a non-zero surface term in a_{2k-1} inevitably appears. We demonstrated that discontinuities of even order derivatives contribute a positive amount to the Casimir energy, while contributions from discontinuities of odd order derivatives are negative.

We have also analysed renormalization of theories with stronger (delta-function) singularities. Our main message is that an independent counterterm to the surface tension is needed in this case. Therefore, the one loop surface tension crucially depends on the normalisation condition. If no such condition is specified, one cannot give a scheme-independent meaning to the surface tension.

One may extend our results by using the heat kernel expansion with non-smooth gauge fields [21], non-smooth geometries [15], other surface singularities [22], and, perhaps, even singularities of conical type [23].

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- [1] K. A. Milton, *The Casimir effect: Physical manifestations of zero-point energy*. World Scientific, River Edge, 2001.
 - [2] M. Bordag, U. Mohideen and V. M. Mostepanenko, *Phys. Rept.* **353**, 1 (2001) [arXiv:quant-ph/0106045].
 - [3] M. Bordag, K. Kirsten and D. Vassilevich, *Phys. Rev. D* **59**, 085011 (1999) [arXiv:hep-th/9811015].
 - [4] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*. Cambridge Univ. Press, Cambridge, 1982.
 - [5] M. Bordag, A. S. Goldhaber, P. van Nieuwenhuizen and D. Vassilevich, *Phys. Rev. D* **66**, 125014 (2002) [arXiv:hep-th/0203066].
 - [6] N. Graham, R. L. Jaffe and H. Weigel, *Int. J. Mod. Phys. A* **17**, 846 (2002) [arXiv:hep-th/0201148].
 - [7] K. A. Milton, *Phys. Rev. D* **68**, 065020 (2003) [arXiv:hep-th/0210081].
 - [8] N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, M. Scandurra and H. Weigel, *Phys. Lett. B* **572**, 196 (2003) [arXiv:hep-th/0207205]; R. L. Jaffe, *AIP Conf. Proc.* **687**, 3 (2003) [arXiv:hep-th/0307014]; N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, O. Schroeder and H. Weigel, *Nucl. Phys. B* **677**, 379 (2004) [arXiv:hep-th/0309130].
 - [9] K. A. Milton, Casimir energies and pressures for delta-function potentials, arXiv:hep-th/0401090; K. A. Milton, Finite Casimir energies in renormalizable quantum field theory, arXiv:hep-th/0401117.
 - [10] J. S. Dowker and R. Critchley, *Phys. Rev. D* **13**, 3224 (1976); S. W. Hawking, *Commun. Math. Phys.* **55**, 133 (1977).

- [11] E. Elizalde, Ten physical applications of spectral zeta functions. Springer, Berlin, 1995; K. Kirsten, Spectral functions in mathematics and physics. Chapman & Hall/CRC, Boca Raton, FL, 2001.
- [12] D. V. Vassilevich, Phys. Rept. **388**, 279 (2003) [arXiv:hep-th/0306138].
- [13] S. A. Fulling, J. Phys. A **36**, 6529 (2003) [arXiv:quant-ph/0302117].
- [14] P. B. Gilkey, Compositio Math. **38**, 201–240 (1979); P. B. Gilkey, Contemp. Math. **73**, 79–85 (1988).
- [15] P. B. Gilkey, K. Kirsten and D. V. Vassilevich, Nucl. Phys. B **601**, 125 (2001) [arXiv:hep-th/0101105].
- [16] A. O. Barvinsky and G. A. Vilkovisky, Nucl. Phys. B **282**, 163 (1987).
- [17] M. Bordag and D. V. Vassilevich, J. Phys. A **32**, 8247 (1999) [arXiv:hep-th/9907076].
- [18] M. Bordag and K. Kirsten, Phys. Rev. D **53**, 5753 (1996) [arXiv:hep-th/9608070]; M. Scandurra, J. Phys. A **32**, 5679 (1999) [arXiv:hep-th/9811164].
- [19] I. Drozdov, Int. J. Mod. Phys. A **18**, 3369 (2003).
- [20] V. N. Marachevsky and D. V. Vassilevich, Nucl. Phys. B **677**, 535 (2004) [arXiv:hep-th/0309019].
- [21] I. G. Moss, Phys. Lett. B **491**, 203 (2000) [arXiv:hep-th/0007185].
- [22] P. Gilkey, K. Kirsten and D. Vassilevich, Lett. Math. Phys. **63**, 29 (2003) [arXiv:hep-th/0208130].
- [23] M. Bordag, K. Kirsten and S. Dowker, Commun. Math. Phys. **182**, 371 (1996) [arXiv:hep-th/9602089].